Reversion Method Applied to Boundary Value Problems

S Baiju *

Abstract

In this paper, the Reversion Method is applied to solve nonlinear boundary value problems with polynomial, exponential and trigonometric nonlinearity. In comparison with the existing techniques, the Reversion method is easy to apply, convergence is rapid, accuracy is better and is good for certain type of nonlinear ordinary differential equations. Numerical results compared with the existing methods show the efficiancy of the method.

Keywords: Reversion Method, Shooting Type Laplace Decomposition Algorithm, Laplace-Adomian Decomposition Algorithm, Adomian Decomposition Metod, Analytic solution and Numerical treatments.

1 Introduction

Analytic solutions of nonlinear boundary value problems are important and is very rare to obtain. Recently many methods like Adomian Decomposition Method (ADM) [1, 7, 14, 20], Homotopy Perturbation Method (HPM) [6, 8, 10], Shooting Type Laplace Decomposition Algoritm (STLADA)[5, 15, 16] have been discussed in detail and their applications have been illustrated. Reversion Method is applicable

**Assistant Professor, Department of Mathematics, Government College, Kariavattom, Thiruvananthapuram, Kerala, E-mail: baijupoint@gmail.com*

to the solution of nonlinear differential equations of a certain class. This method is introduced by Louis. A. Pipes in 1952 [11, 13, 18]. Other procedures leading to the same general ideas were published in [12]. These papers deal with the nonlinear ordinary differential equations with initial value problems. But, in the present paper, the Reversion Method is applied to solve some nonlinear boundary value problems of any order.

In the study of certain physical problems arising in mechanics especially fluid mechanics, thermodynamics and many biological phenomena whose mathematical formulation leads to nonlinear differential equations. It has been foud that the solution of such problems is very difficult to obtain or much labor is required to solve if the nonlinearity is complicated. Complexity of the available methods reveal the need of the Reversion method at the present stage. This method is applicable to the initial value problems and boundary value problems with polynomial, exponential and trigonometric nonlinearity. Most of the available methods can't be applied to trigonometric nonlinearity. But the Reversion Method is applicable to these type of nonlinearities. The Reversion Method reduces the nonlinear differential equations to a system of linear differential equations. When the methods of the Laplace transform theory or the method of linear differential equation theory are applied to the solution of the resulting system of differential equations, the method saves a great deal of numerical effort. In the Reversion Method, there is no need of choosing the initial term and the initial term obtained directly from the method.

In the present paper, I introduce the application of the Reversion Method (RM) to nonlinear boundary value problems, to ordinary differential equations with different type of nonlinearities. This is an easy method to solve such type of nonlinear differential equations. Numerical experiments are very encouraging.

2 Analysis of the Method

Consider the nonlinear differential equation of the form

$$a_1u + a_2u^2 + a_3u^3 + a_4u^4 + ... + a_nu^n + ... = kf(t) \tag{2.1}$$

where t is in the independent variable, u is the dependent variable which is to be determined, k is a constant, $f(t)$ is a given function in t. The coefficient a_i's are the operators with $a_1 \neq 0$ and the initial or boundary conditions are specified.

Let the solution of (2.1) is of the form

$$u = A_1k + A_2k^2 + A_3k^3 + ... \tag{2.2}$$

the terms A_i's: $i = 1, 2, 3, ...$ are to be determined. Substituting (2.2) in (2.1) and equating coefficients of like powers of k. Then we get the following equations for the coefficients.

$$A_1 = \frac{f(t)}{a_1} \tag{2.3}$$

$$A_2 = \frac{-a_2A_1^2}{a_1} \tag{2.4}$$

$$A_3 = \frac{-1}{a_1}[2a_2A_1A_2 + a_3A_1^3] \tag{2.5}$$

$$A_4 = \frac{-1}{a_1}[a_2(A_2^2 + 2A_1A_3) + 3a_3A_1^2A_2 + a_4A_1^4] \tag{2.6}$$

$$A_5 = \frac{-1}{a_1}[2a_2(A_1A_+A_2A_3) + 3a_3(A_1A_2^2 + A_1^2A_3) + 4a_4A_2A_1^3 + a_5A_1^5] \tag{2.7}$$

$$A_6 = \frac{-1}{a_1}[a_2(A_3^2+2A_1A_5+2A_2A_4)+a_3(A_2^3 3A_1^2A_4+6A_1A_2A_3)+2a_4(2A_1^2A_2^2+2A_1^3A_3)+ 5a_5A_1^4A_2 + a_6A_1^6] \tag{2.8}$$

Additional coefficients can be found in Van Orstrand's paper [17] in which a list of first thirteen coefficients is given.

For convenience, we take $k = 1$ and on substitution into (2.2), we get the approximate analytic solution. All the coefficients obtained are linear diffrerential equations with initial or boundary conditios, which can be solved either by using Laplace transforms or any method suitable to the linear differential equation theory. Then we obtain the solutions of the system of linear equations as A_i's.

3 Numerical Experiments

Example 3.1(Polynomial nonlinearity)

Consider the nonlinear differential equation of the electrical behavior of cell membrane [2, 3, 4, 9, 14, 20]

$$\frac{d^2}{dx^2} = u + u^3 \tag{3.1}$$

with boundary conditions $u(0) = 1, u'(1) = 0; 0 < x < 1$ (3.2)

Comparison of (3.1) with (2.1), we see that

$a_1 = \frac{d^2}{dx^2} - 1$, $a_2 = 0$, $a_3 = -1$, $a_4 = a_5 = ... = 0$ and $f(t) = 0$

Assume that $u(x) = A_1k + A_2k^2 + A_3k^3 + ...$ is a solution of (3.1).

By Reversion method, $A_1 = \frac{f(t)}{a_1}$ which yields the differential equation as

$$\frac{d^2A_1}{dx^2} - A_1 = 0 \tag{3.3}$$

Solving (3.3), we get

$$A_1(x) = c_1e^x + c_2e^{-x} \tag{3.4}$$

Hence the first approximation is $y(x) = A_1k$

For convenience $k = 1$, the approximate solution $y(x) = c_1e^x + c_2e^{-x}$ (3.5)

Using the boundary conditions from (3.2) to (3.5), we get the system of equations as

$$c_1 + c_2 = 1; c_1e - c_2e^{-1} = 1 \tag{3.6}$$

Solving these system, we get $c_1 = \frac{1}{e^2+1}$ and $c_2 = \frac{e^2}{e^2+1}$.

Hence $A_1(x) = \frac{1}{e^2+1}e^x + \frac{e^2}{e^2+1}e^{-x}$ and the solution for the first approximation is

$$y(x) = \frac{1}{e^2+1}e^x + \frac{e^2}{e^2+1}e^{-x} \tag{3.7}$$

For the second approximation, $A_2 = \frac{-a_2A_1^2}{a_1}$ gives $a_1A_2 = 0$

Corresponding linear differential equation is $\frac{d^2A_2}{dx^2} - A_2 = 0$, gives the solution as

$A_2(x) = c_3e^x + c_4e^{-x}$ and the approximate solution at this stage is

$$y(x) = \frac{1}{e^2+1}e^x + \frac{e^2}{e^2+1}e^{-x} + c_3e^x + c_4e^{-x} \tag{3.8}$$

Using the boundary conditions, we get the values of c_3 and c_4. Hence $A_2 = 0$ and the approximate solution is same at this stage. For third approximation, $A_3 = \frac{-1}{a_1}[2a_2A_1A_2 + a_3A_1^3]$, then $a_1A_3 = -a_3A_1^3$. On substitution of the values, we get a linear differential equation as $\frac{d^2A_3}{dx^2} - A_3 = [\frac{1}{e^2+1}e^x + \frac{e^2}{e^2+1}e^{-x}]^3$ yielding the solution as

$$A_3(x) = c_5e^x + c_6e^{-x} + \frac{1}{(e^2+1)^3}[\frac{e^{3x}}{8} + \frac{e^{6-3x}}{8} + \frac{3}{2}xe^{2+x} - \frac{3}{2}xe^{4-x}] \tag{3.9}$$

At this stage the approximate solution is

$$y(x) = \frac{1}{e^2+1}e^x + \frac{e^2}{e^2+1}e^{-x} + c_5e^x + c_6e^{-x} + \frac{1}{(e^2+1)^3}[\frac{e^{3x}}{8} + \frac{e^{6-3x}}{8} + \frac{3}{2}xe^{2+x} - \frac{3}{2}xe^{4-x}]$$

Using the conditions in (3.2), one can find the parameters c_5 and c_6, the approximate analytic solution obtained at this stage as

$$u(x) = \frac{1}{e^2+1}e^x + \frac{e^2}{e^2+1}e^{-x} - \frac{e^6+24e^4+1}{8(e^2+1^4}e^x - \frac{e^2(e^6-24e^2+1)}{8(e^2+1)^4}e^{-x} + \frac{1}{(e^2+1)^3}[\frac{e^{3x}}{8} + \frac{e^{6-3x}}{8} + \frac{3}{2}xe^{2+x} - \frac{3}{2}xe^{4-x}] \tag{3.10}$$

The above process is applied to (2.6), we get $A_4 = 0$. It can also be seen that

$$A_6 = A_8 = ... = A_{2n} = 0.$$

For the next approximation the analytic solution is

$$u(x) = \frac{1}{e^2+1}e^x + \frac{e^2}{e^2+1}e^{-x} - \frac{e^6+24e^4+1}{8(e^2+1^4}e^x - \frac{e^2(e^6-24e^2+1)}{8(e^2+1)^4}e^{-x} + \frac{1}{(e^2+1)^3}[\frac{e^{3x}}{8} + \frac{e^{6-3x}}{8} + \frac{3}{2}xe^{2+x} - \frac{3}{2}xe^{4-x}] + \frac{2e^{12}+92e^{10}+933e^8-738e^6+165e^4+20e^2+2}{64(e^2+1)^7}e^x + \frac{e^2(2-978e^6+837e^4+20e^{10}-52e^2+2e^{12}-123e^8)}{(e^2+1)^7}e^{-x} - \frac{3}{64}(\frac{e^6+24e^4+1}{(e^2+1)^6})e^{3x} - \frac{3e^2(e^6-24e^2+1)}{16(e^2+1)^6}xe^x + \frac{1}{64(e^2+1)^5}e^{5x} - \frac{3e^6}{16(e^2+1)^5}xe^{-x} + \frac{9e^2}{64(e^2+1)^5}(4x-3)e^{3x} - \frac{9e^4}{8(e^2+1)^5}(x^2-x)e^x + \frac{3(e^6+24e^4+1)}{16(e^2+1)^6}xe^{4-x} - \frac{3e^2(e^6-24e^2+1)}{64(e^2+1)^6}e^{4-x} + \frac{3e^4}{16(e^2+1)^5}xe^x + \frac{1}{64(e^2+1)^5}e^{10-5x} - \frac{9e^6}{8(e^2+1)^5}e^{-x}(x^2+x) - \frac{9e^8}{64(e^2+1)^5}e^{-3x}(4x+3) - \frac{3e^2(e^6+24e^4+1)}{8(e^2+1)^6}xe^x + \frac{3e^4(e^6-24e^2+1)}{8(e^2+1)^6}xe^{-x} + \frac{3e^2}{32(e^2+1)^5}e^{3x} + \frac{3e^8}{32(e^2+1)^5}e^{-3x} + \frac{9e^4}{4(e^2+1)^5}e^x(x^2-x) + \frac{9e^6}{4(e^2+1)^5}(x^2+x)e^{-x} \tag{3.11}$$

Value x	RM S_2	STLADA S_4	ADM S_5	Var [3]	Upper [19]	Lower [19]	Exact
0.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.1	-0.0414	-0.0414	-0.0414	-0.4041	-0.0415	-0.0414	-0.0414
0.2	-0.0733	-0.0733	-0.0733	-0.0730	-0.0733	-0.0732	-0.0733
0.3	-0.0958	-0.0958	-0.0958	-0.0970	-0.0959	-0.0958	-0.0958
0.4	-0.1093	-0.1092	-0.1092	-0.1116	-0.1093	-0.1092	-0.1092
0.5	-0.1137	-0.1137	-0.1137	-0.1165	-0.1138	-0.1137	-0.1137

Table 2: **Comparison with the existing results**

stage with $k = 1$ is

$$u(x) = \frac{1}{e+1}e^x + \frac{e}{e+1}e^{-x} - 1 + \frac{5e^2+15e+2}{6(e+1)^3}e^x + \frac{e(2e^2+9e-1)}{6(e+1)^3}e^{-x} + \frac{1}{6(e+1)^2}[e^{2x} - 12e - 3xe^{x+1} - 3xe^x + e^{2-2x} + 3xe^{2-x} + 3xe^{1-x} - 3e^2 - 3]$$

Exact solution of the nonlinear differential equation is $u(x) = -ln2+ln\{n.sec[c(\frac{x-0.5}{2})]\}$ where c is a root of $\sqrt{2} = c.sec(\frac{c}{4})$ lying between 0 and $\frac{\pi}{2}$; namely 1.3360557. Table 2 shows the convergence of the sequences $S_n(x)$ for various values of x; $n = 1, 2, 3, 4, ...$ From this, one can observe that the Reversion Method converges very fast for lower number of iterations. Comparison is made with the existing results using Shooting Type Laplace Decomposition Algorithm (STLADA) [5], Adomian Decomposition Method (ADM) [14, 20], Complementary Variational Method [3] and Loweer and upper solutions [19]. From these methods, Reversion Method converges very rapidly to the exact solution.

4 Conclusion

In the present method, we see that convergence is very fast and give accurate results. This method is closely associated with the method of Picard's for solving differential equations. A study of the convergence of the process indicates that convergence is rapid if the nonlinearities involved are small and in such cases it is only necessary to compute a few terms of the series to obtain an accurate solution of the differential equations under consideration. Computations involved are performed using Maple.

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